Asymptotic inference for a stochastic differential equation with uniformly distributed time delay

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Abstract

For the affine stochastic delay differential equation

$$\mathrm{d}X(t) = a \int_{-1}^{0} X(t+u)\,\mathrm{d}u\,\mathrm{d}t + \mathrm{d}W(t), \qquad t \geqslant 0,$$

the local asymptotic properties of the likelihood function are studied. Local asymptotic normality is proved in case of $a \in \left(-\frac{\pi^2}{2}, 0\right)$, local asymptotic mixed normality is shown if $a \in (0, \infty)$, periodic local asymptotic mixed normality is valid if $a \in \left(-\infty, -\frac{\pi^2}{2}\right)$, and only local asymptotic quadraticity holds at the points $-\frac{\pi^2}{2}$ and 0. Applications to the asymptotic behaviour of the maximum likelihood estimator $\widehat{a}_T$ of a based on $(X(t))_{t\in[0,T]}$ are given as $T \to \infty$.

1 Introduction

Assume $(W(t))_{t\in\mathbb{R}_+}$ is a standard Wiener process, $a \in \mathbb{R}$, and $(X^{(a)}(t))_{t\in\mathbb{R}_+}$ is a solution of the affine stochastic delay differential equation (SDDE)

$$(1.1) \qquad \begin{cases} \mathrm{d}X(t) = a \int_{-1}^{0} X(t+u)\,\mathrm{d}u\,\mathrm{d}t + \mathrm{d}W(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-1, 0], \end{cases}$$

where $(X_0(t))_{t\in[-1,0]}$ is a continuous stochastic process independent of $(W(t))_{t\in\mathbb{R}_+}$. The SDDE (1.1) can also be written in the integral form

$$(1.2) \qquad \begin{cases} X(t) = X_0(0) + a \int_0^t \int_{-1}^{0} X(s+u)\,\mathrm{d}u\,\mathrm{d}s + W(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-1, 0]. \end{cases}$$

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Equation (1.1) is a special case of the affine stochastic delay differential equation

$$(1.3) \qquad \begin{cases} \mathrm{d}X(t) = \int_{-r}^{0} X(t+u)\, m_\theta(\mathrm{d}u)\, \mathrm{d}t + \mathrm{d}W(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-r, 0], \end{cases}$$

where $r > 0$, and for each $\theta \in \Theta$, m_θ, is a finite signed measure on $[-r, 0]$ see Gushchin and Küchler [3]. In that paper local asymptotic normality has been proved for stationary solutions. In Gushchin and Küchler [1], the special case of (1.3) has been studied with $r = 1$, $\Theta = \mathbb{R}^2$, and $m_\theta = a\delta_0 + b\delta_{-1}$ for $\theta = (a, b)$, where δ_x denotes the Dirac measure concentrated at $x \in \mathbb{R}$, and they described the local properties of the likelihood function for the whole parameter space $\mathbb{R}^2$.

The solution $(X^{(a)}(t))_{t\in\mathbb{R}_+}$ of (1.1) exists, is pathwise uniquely determined and can be represented as

$$(1.4) \qquad X^{(a)}(t) = x_{0,a}(t)X_0(0) + a\int_{-1}^{0}\int_{u}^{0} x_{0,a}(t+u-s)X_0(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t x_{0,a}(t-s)\,\mathrm{d}W(s),$$

for $t \in \mathbb{R}_+$, where $(x_{0,a}(t))_{t\in[-1,\infty)}$ denotes the so-called fundamental solution of the deterministic homogeneous delay differential equation

$$(1.5) \qquad \begin{cases} x(t) = x_0(0) + a\int_0^t \int_{-1}^0 x(s+u)\,\mathrm{d}u\,\mathrm{d}s, & t \in \mathbb{R}_+, \\ x(t) = x_0(t), & t \in [-1, 0]. \end{cases}$$

with initial function

$$x_0(t) := \begin{cases} 0, & t \in [-1, 0), \\ 1, & t = 0. \end{cases}$$

In the trivial case of $a = 0$, we have $x_{0,0}(t) = 1$, $t \in \mathbb{R}_+$, and $X^{(0)}(t) = X_0(0) + W(t)$, $t \in \mathbb{R}_+$. In case of $a \in \mathbb{R} \setminus \{0\}$, the behaviour of $(x_{0,a}(t))_{t\in[-1,\infty)}$ is connected with the so-called characteristic function $h_a : \mathbb{C} \to \mathbb{C}$, given by

$$(1.6) \qquad h_a(\lambda) := \lambda - a\int_{-1}^{0} \mathrm{e}^{\lambda u}\,\mathrm{d}u, \qquad \lambda \in \mathbb{C},$$

and the set Λ_a of the (complex) solutions of the so-called characteristic equation for (1.5),

$$(1.7) \qquad \lambda - a\int_{-1}^{0} \mathrm{e}^{\lambda u}\,\mathrm{d}u = 0.$$

Note that a complex number λ solves (1.7) if and only if $(\mathrm{e}^{\lambda t})_{t\in[-1,\infty)}$ solves (1.5) with initial function $x_0(t) = \mathrm{e}^{\lambda t}$, $t \in [-1, 0]$. Applying usual methods (e.g., argument principle in complex analysis and the existence of local inverses of holomorphic functions), one can derive the following properties of the set Λ_a, see, e.g., Reiß [8]. We have $\Lambda_a \neq \emptyset$, and Λ_a consists of isolated points. Moreover, Λ_a is countably infinite, and for each $c \in \mathbb{R}$, the set $\{\lambda \in \Lambda_a : \mathrm{Re}(\lambda) \geqslant c\}$ is finite. In particular,

$$v_0(a) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \Lambda_a\} < \infty.$$

Put

$$v_1(a) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \Lambda_a,\, \mathrm{Re}(\lambda) < v_0(a)\},$$

where $\sup\emptyset := -\infty$. We have the following cases:

(i) If $a \in \left(-\frac{\pi^2}{2}, 0\right)$ then $v_0(a) < 0$;

(ii) If $a = -\frac{\pi^2}{2}$ then $v_0(a) = 0$ and $v_0(a) \notin \Lambda_a$;

(iii) If $a \in \left(-\infty, -\frac{\pi^2}{2}\right)$ then $v_0(a) > 0$ and $v_0(a) \notin \Lambda_a$;

(iv) If $a \in (0, \infty)$ then $v_0(a) > 0$, $v_0(a) \in \Lambda_a$, $m(v_0(a)) = 1$ (where $m(v_0(a))$ denotes the multiplicity of $v_0(a)$), and $v_1(a) < 0$.

For any $\gamma > v_0(a)$, we have $x_{0,a}(t) = \mathrm{O}(\mathrm{e}^{\gamma t})$, $t \in \mathbb{R}_+$. In particular, $(x_{0,a}(t))_{t\in\mathbb{R}_+}$ is square integrable if (and only if, see Gushchin and Küchler [2]) $v_0(a) < 0$. The Laplace transform of $(x_{0,a}(t))_{t\in\mathbb{R}_+}$ is given by

$$\int_0^\infty \mathrm{e}^{-\lambda t} x_{0,a}(t)\,\mathrm{d}t = \frac{1}{h_a(\lambda)}, \qquad \lambda \in \mathbb{C}, \qquad \mathrm{Re}(\lambda) > v_0(a).$$

Based on the inverse Laplace transform and Cauchy's residue theorem, the following crucial lemma can be shown (see, e.g., Gushchin and Küchler [1, Lemma 1.1]).

1.1 Lemma. *For each* $a \in \mathbb{R} \setminus \{0\}$ *and each* $c \in (-\infty, v_0(a))$, *there exists* $\gamma \in (-\infty, c)$ *such that the fundamental solution* $(x_{0,a}(t))_{t\in[-1,\infty)}$ *of* (1.5) *can be represented in the form*

$$x_{0,a}(t) = \psi_{0,a}(t)\mathrm{e}^{v_0(a)t} + \sum_{\substack{\lambda\in\Lambda_a \\ \mathrm{Re}(\lambda)\in[c,v_0(a))}} c_a(\lambda)\mathrm{e}^{\lambda t} + \mathrm{o}(\mathrm{e}^{\gamma t}), \qquad \textit{as } t \to \infty,$$

with some constants $c_a(\lambda)$, $\lambda \in \Lambda_a$, *and with*

$$\psi_{0,a}(t) := \begin{cases} \dfrac{v_0(a)}{v_0(a)^2 + 2v_0(a) - a}, & \textit{if } v_0(a) \in \Lambda_a \textit{ and } m(v_0(a)) = 1, \\ A_0(a)\cos(\kappa_0(a)t) + B_0(a)\sin(\kappa_0(a)t) & \textit{if } v_0(a) \notin \Lambda_a, \end{cases}$$

with $\kappa_0(a) := |\,\mathrm{Im}(\lambda_0(a))|$, *where* $\lambda_0(a) \in \Lambda_a$ *is given by* $\mathrm{Re}(\lambda_0(a)) = v_0(a)$, *and*

$$A_0(a) := \frac{2[(v_0(a)^2 - \kappa_0(a)^2)(v_0(a) - 2) - av_0(a)]}{(v_0(a)^2 - \kappa_0(a)^2 + 2v_0(a) - a)^2 + 4\kappa_0(a)^2(v_0(a) + 1)^2},$$

$$B_0(a) := \frac{2(v_0(a)^2 + \kappa_0(a)^2 + a)\kappa_0(a)}{(v_0(a)^2 - \kappa_0(a)^2 + 2v_0(a) - a)^2 + 4\kappa_0(a)^2(v_0(a) + 1)^2}.$$

2 Quadratic approximations to likelihood ratios

We recall some definitions and statements concerning quadratic approximations to likelihood ratios based on Jeganathan [5], Le Cam and Yang [6] and van der Vaart [9].

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $\Theta \subset \mathbb{R}^p$ be an open set. For each $\boldsymbol{\theta} \in \Theta$, let $(X^{(\boldsymbol{\theta})}(t))_{t\in[-1,\infty)}$ be a continuous stochastic process on $(\Omega, \mathcal{A}, \mathbb{P})$. For each $T \in \mathbb{R}_+$, let $\mathbb{P}_{\boldsymbol{\theta},T}$ be the probability measure induced by $(X^{(\boldsymbol{\theta})}(t))_{t\in[-1,T]}$ on the space $(C([-1,T]), \mathcal{B}(C([-1,T])))$.

2.1 Definition. *The family* $(C([-1,T]), \mathcal{B}(C([-1,T])), \{\mathbb{P}_{\boldsymbol{\theta},T} : \boldsymbol{\theta} \in \Theta\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments is said to have locally asymptotically quadratic (LAQ) likelihood ratios at* $\boldsymbol{\theta} \in \Theta$ *if there exist (scaling) matrices* $\boldsymbol{r}_{\boldsymbol{\theta},T} \in \mathbb{R}^{p\times p}$, $T \in \mathbb{R}_{++}$, *random vectors* $\boldsymbol{\Delta}_{\boldsymbol{\theta}} : \Omega \to \mathbb{R}^p$ *and* $\boldsymbol{\Delta}_{\boldsymbol{\theta},T} : \Omega \to \mathbb{R}^p$, $T \in \mathbb{R}_{++}$, *and random matrices* $\boldsymbol{J}_{\boldsymbol{\theta}} : \Omega \to \mathbb{R}^{p\times p}$ *and* $\boldsymbol{J}_{\boldsymbol{\theta},T} : \Omega \to \mathbb{R}^{p\times p}$, $T \in \mathbb{R}_{++}$, *such that*

$$\log \frac{\mathrm{d}\mathbb{P}_{\boldsymbol{\theta}+\boldsymbol{r}_{\boldsymbol{\theta},T}\boldsymbol{h}_T,T}}{\mathrm{d}\mathbb{P}_{\boldsymbol{\theta},T}}(X^{(\boldsymbol{\theta})}|_{[-1,T]}) = \boldsymbol{h}_T^\top \boldsymbol{\Delta}_{\boldsymbol{\theta},T} - \frac{1}{2}\boldsymbol{h}_T^\top \boldsymbol{J}_{\boldsymbol{\theta},T}\boldsymbol{h}_T + \mathrm{o}_{\mathbb{P}}(1) \qquad \text{as } T \to \infty \tag{2.1}$$

whenever $\boldsymbol{h}_T \in \mathbb{R}^p$, $T \in \mathbb{R}_{++}$, *is a bounded family satisfying* $\boldsymbol{\theta} + \boldsymbol{r}_{\boldsymbol{\theta},T}\boldsymbol{h}_T \in \Theta$ *for all* $T \in \mathbb{R}_{++}$,

$$(\boldsymbol{\Delta}_{\boldsymbol{\theta},T}, \boldsymbol{J}_{\boldsymbol{\theta},T}) \xrightarrow{\mathcal{D}} (\boldsymbol{\Delta}_{\boldsymbol{\theta}}, \boldsymbol{J}_{\boldsymbol{\theta}}) \qquad \text{as } T \to \infty, \tag{2.2}$$

and we have

$$\mathbb{P}\left(\boldsymbol{J}_{\boldsymbol{\theta}} \text{ is symmetric and strictly positive definite}\right) = 1 \tag{2.3}$$

and

$$\mathbb{E}\left(\exp\left\{\boldsymbol{h}^\top \boldsymbol{\Delta}_{\boldsymbol{\theta}} - \frac{1}{2}\boldsymbol{h}^\top \boldsymbol{J}_{\boldsymbol{\theta}}\boldsymbol{h}\right\}\right) = 1, \qquad \boldsymbol{h} \in \mathbb{R}^p. \tag{2.4}$$

2.2 Definition. *A family* $(C([-1,T]), \mathcal{B}(C([-1,T])), \{\mathbb{P}_{\boldsymbol{\theta},T} : \boldsymbol{\theta} \in \Theta\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments is said to have locally asymptotically mixed normal (LAMN) likelihood ratios at* $\boldsymbol{\theta} \in \Theta$ *if it is LAQ at* $\boldsymbol{\theta} \in \Theta$, *and the conditional distribution of* $\boldsymbol{\Delta}_{\boldsymbol{\theta}}$ *given* $\boldsymbol{J}_{\boldsymbol{\theta}}$ *is* $\mathcal{N}_p(\boldsymbol{0}, \boldsymbol{J}_{\boldsymbol{\theta}})$, *or, equivalently, there exist a random vector* $\mathcal{Z} : \Omega \to \mathbb{R}^p$ *and a random matrix* $\eta_{\boldsymbol{\theta}} : \Omega \to \mathbb{R}^{p\times p}$, *such that they are independent,* $\mathcal{Z} \stackrel{\mathcal{D}}{=} \mathcal{N}_p(\boldsymbol{0}, \boldsymbol{I}_p)$, *and* $\boldsymbol{\Delta}_{\boldsymbol{\theta}} = \eta_{\boldsymbol{\theta}}\mathcal{Z}$, $\boldsymbol{J}_{\boldsymbol{\theta}} = \eta_{\boldsymbol{\theta}}\eta_{\boldsymbol{\theta}}^\top$.

2.3 Definition. *The family* $(C([-1,T]), \mathcal{B}(C([-1,T])), \{\mathbb{P}_{\boldsymbol{\theta},T} : \boldsymbol{\theta} \in \Theta\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments is said to have periodic locally asymptotically mixed normal (PLAMN) likelihood ratios at* $\boldsymbol{\theta} \in \Theta$ *if there exist* $D \in \mathbb{R}_{++}$, *(scaling) matrices* $\boldsymbol{r}_{\boldsymbol{\theta},T} \in \mathbb{R}^{p\times p}$, $T \in \mathbb{R}_{++}$, *random vectors* $\boldsymbol{\Delta}_{\boldsymbol{\theta}}(d) : \Omega \to \mathbb{R}^p$, $d \in [0, D)$, *and* $\boldsymbol{\Delta}_{\boldsymbol{\theta},T} : \Omega \to \mathbb{R}^p$, $T \in \mathbb{R}_{++}$, *and random matrices* $\boldsymbol{J}_{\boldsymbol{\theta}}(d) : \Omega \to \mathbb{R}^{p\times p}$, $d \in [0, D)$, *and* $\boldsymbol{J}_{\boldsymbol{\theta},T} : \Omega \to \mathbb{R}^{p\times p}$, $T \in \mathbb{R}_{++}$, *such that* (2.1) *holds whenever* $\boldsymbol{h}_T \in \mathbb{R}^p$, $T \in \mathbb{R}_{++}$, *is a bounded family satisfying* $\boldsymbol{\theta} + \boldsymbol{r}_{\boldsymbol{\theta},T}\boldsymbol{h}_T \in \Theta$ *for all* $T \in \mathbb{R}_{++}$,

$$(\boldsymbol{\Delta}_{\boldsymbol{\theta},kD+d}, \boldsymbol{J}_{\boldsymbol{\theta},kD+d}) \xrightarrow{\mathcal{D}} (\boldsymbol{\Delta}_{\boldsymbol{\theta}}(d), \boldsymbol{J}_{\boldsymbol{\theta}}(d)) \qquad \text{as } k \to \infty \tag{2.5}$$

for all $d \in [0, D)$, *we have*

$$\mathbb{P}\left(\boldsymbol{J}_{\boldsymbol{\theta}}(d) \textit{ is symmetric and strictly positive definite}\right) = 1, \qquad d \in [0, D), \tag{2.6}$$

and for each $d \in [0, D)$, *the conditional distribution of* $\boldsymbol{\Delta}_{\boldsymbol{\theta}}(d)$ *given* $\boldsymbol{J}_{\boldsymbol{\theta}}(d)$ *is* $\mathcal{N}_p(\mathbf{0}, \boldsymbol{J}_{\boldsymbol{\theta}}(d))$, *or, equivalently, there exist a random vector* $\mathcal{Z} : \Omega \to \mathbb{R}^p$ *and a random matrix* $\eta_{\boldsymbol{\theta}}(d) : \Omega \to \mathbb{R}^{p \times p}$ *such that they are independent,* $\mathcal{Z} \stackrel{\mathcal{D}}{=} \mathcal{N}_p(\mathbf{0}, \boldsymbol{I}_p)$, *and* $\boldsymbol{\Delta}_{\boldsymbol{\theta}}(d) = \eta_{\boldsymbol{\theta}}(d)\mathcal{Z}$, $\boldsymbol{J}_{\boldsymbol{\theta}}(d) = \eta_{\boldsymbol{\theta}}(d)\eta_{\boldsymbol{\theta}}^\top(d)$.

2.4 Remark. The notion of LAMN is defined in Le Cam and Yang [6] and Jeganathan [5] so that PLAMN in the sense of Definiton 2.3 is LAMN as well.

2.5 Definition. *A family* $(C([-1,T]), \mathcal{B}(C([-1,T])), \{\mathbb{P}_{\boldsymbol{\theta},T} : \boldsymbol{\theta} \in \Theta\})_{T \in \mathbb{R}_{++}}$ *of statistical experiments is said to have locally asymptotically normal (LAN) likelihood ratios at* $\boldsymbol{\theta} \in \Theta$ *if it is LAMN at* $\boldsymbol{\theta} \in \Theta$, *and* $\boldsymbol{J}_{\boldsymbol{\theta}}$ *is deterministic.*

3 Radon–Nikodym derivatives

From this section, we will consider the SDDE (1.1) with fixed continuous initial process $(X_0(t))_{t \in [-1,0]}$. Further, for all $T \in \mathbb{R}_{++}$, let $\mathbb{P}_{a,T}$ be the probability measure induced by $(X^{(a)}(t))_{t \in [-1,T]}$ on $(C([-1,T]), \mathcal{B}(C([-1,T])))$. In order to calculate Radon–Nikodym derivatives $\frac{\mathrm{d}\mathbb{P}_{\widetilde{a},T}}{\mathrm{d}\mathbb{P}_{a,T}}$ for certain $a, \widetilde{a} \in \mathbb{R}$, we need the following statement, which can be derived from formula (7.139) in Section 7.6.4 of Liptser and Shiryaev [7].

3.1 Lemma. *Let* $a, \widetilde{a} \in \mathbb{R}$. *Then for all* $T \in \mathbb{R}_{++}$, *the measures* $\mathbb{P}_{a,T}$ *and* $\mathbb{P}_{\widetilde{a},T}$ *are absolutely continuous with respect to each other, and*

$$\begin{aligned}
&\log \frac{\mathrm{d}\mathbb{P}_{\widetilde{a},T}}{\mathrm{d}\mathbb{P}_{a,T}}(X^{(a)}|_{[-1,T]}) \\
&= (\widetilde{a} - a) \int_0^T \int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\,\mathrm{d}X^{(a)}(t) - \frac{\widetilde{a}^2 - a^2}{2} \int_0^T \left(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u \right)^2 \mathrm{d}t \\
&= (\widetilde{a} - a) \int_0^T \int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\,\mathrm{d}W(t) - \frac{(\widetilde{a} - a)^2}{2} \int_0^T \left(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u \right)^2 \mathrm{d}t.
\end{aligned}$$

In order to investigate convergence of the family

$$(\mathcal{E}_T)_{T \in \mathbb{R}_{++}} := \big(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)), \{\mathbb{P}_{a,T} : a \in \mathbb{R}\}\big)_{T \in \mathbb{R}_{++}} \tag{3.1}$$

of statistical experiments, we derive the following corollary.

3.2 Corollary. *For each* $a \in \mathbb{R}$, $T \in \mathbb{R}_{++}$, $r_{a,T} \in \mathbb{R}$ *and* $h_T \in \mathbb{R}$, *we have*

$$\log \frac{\mathrm{d}\mathbb{P}_{a + r_{a,T} h_T, T}}{\mathrm{d}\mathbb{P}_{a,T}}(X^{(a)}|_{[-1,T]}) = h_T \Delta_{a,T} - \frac{1}{2} h_T^2 J_{a,T},$$

with

$$\Delta_{a,T} := r_{a,T} \int_0^T \int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\,\mathrm{d}W(t), \qquad J_{a,T} := r_{a,T}^2 \int_0^T \left(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u \right)^2 \mathrm{d}t.$$

Consequently, the quadratic approximation (2.1) *is valid.*

4 Local asymptotics of likelihood ratios

4.1 Proposition. *If* $a \in \left(-\frac{\pi^2}{2}, 0\right)$ *then the family* $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ *of statistical experiments given in* (3.1) *is LAN at* a *with scaling* $r_{a,T} = \frac{1}{\sqrt{T}}$, $T \in \mathbb{R}_{++}$, *and with*

$$J_a = \int_0^\infty \left(\int_{-1}^0 x_{0,a}(t+u)\,\mathrm{d}u \right)^2 \mathrm{d}t.$$

4.2 Proposition. *The family* $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ *of statistical experiments given in* (3.1) *is LAQ at* 0 *with scaling* $r_{0,T} = \frac{1}{T}$, $T \in \mathbb{R}_{++}$, *and with*

$$\Delta_0 = \int_0^1 \mathcal{W}(t)\,\mathrm{d}\mathcal{W}(t), \qquad J_0 = \int_0^1 \mathcal{W}(t)^2\,\mathrm{d}t,$$

where $(\mathcal{W}(t))_{t \in [0,1]}$ *is a standard Wiener process.*

4.3 Proposition. *The family* $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ *of statistical experiments given in* (3.1) *is LAQ at* $-\frac{\pi^2}{2}$ *with scaling* $r_{-\frac{\pi^2}{2},T} = \frac{1}{T}$, $T \in \mathbb{R}_{++}$, *and with*

$$\Delta_{-\frac{\pi^2}{2}} = \frac{16 \int_0^1 (\mathcal{W}_1(s)\,\mathrm{d}\mathcal{W}_2(s) - \mathcal{W}_2(s)\,\mathrm{d}\mathcal{W}_1(s)) - 4\pi \int_0^1 (\mathcal{W}_1(s)\,\mathrm{d}\mathcal{W}_1(s) + \mathcal{W}_2(s)\,\mathrm{d}\mathcal{W}_2(s))}{\pi(\pi^2+16)},$$

$$J_{-\frac{\pi^2}{2}} = \frac{16}{\pi^2(\pi^2+16)} \int_0^1 (\mathcal{W}_1(t)^2 + \mathcal{W}_2(t)^2)\,\mathrm{d}t,$$

where $(\mathcal{W}_1(t), \mathcal{W}_2(t))_{t \in [0,1]}$ *is a 2-dimensional standard Wiener process.*

4.4 Proposition. *If* $a \in (0, \infty)$ *then the family* $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ *of statistical experiments given in* (3.1) *is LAMN at* a *with scaling* $r_{a,T} = \mathrm{e}^{-v_0(a)T}$, $T \in \mathbb{R}_{++}$, *and with*

$$\Delta_a = Z\sqrt{J_a}, \qquad J_a = \frac{(1-\mathrm{e}^{-v_0(a)})^2}{2v_0(a)(v_0(a)^2 + 2v_0(a) - a)^2} (U^{(a)})^2,$$

with

$$U^{(a)} = X_0(0) + a \int_{-1}^0 \int_u^0 \mathrm{e}^{-v_0(a)(s-u)} X_0(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^\infty \mathrm{e}^{-v_0(a)s}\,\mathrm{d}W(s),$$

and Z *is a standard normally distributed random variable independent of* J_a.

4.5 Proposition. *If* $a \in \left(-\infty, -\frac{\pi^2}{2}\right)$ *then the family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments given in* (3.1) *is PLAMN at* a *with period* $D = \frac{\pi}{\kappa_0(a)}$, *with scaling* $r_{a,T} = \mathrm{e}^{-v_0(a)T}$, $T \in \mathbb{R}_{++}$, *and with*

$$\Delta_a(d) = Z\sqrt{J_a(d)}, \qquad J_a(d) = \int_0^\infty \mathrm{e}^{-2v_0(a)s}(V^{(a)}(d-s))^2\,\mathrm{d}s, \qquad d \in \left[0, \frac{\pi}{\kappa_0(a)}\right),$$

where

$$V^{(a)}(t) := X_0(0)\varphi_a(t) + a\int_{-1}^0\int_u^0 \varphi_a(t+u-s)\mathrm{e}^{-v_0(a)(s-u)}X_0(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^\infty \varphi_a(t-s)\mathrm{e}^{-v_0(a)s}\,\mathrm{d}W(s), \qquad t \in \mathbb{R}_+,$$

with

$$\varphi_a(t) := A_0(a)\cos(\kappa_0(a)t) + B_0(a)\sin(\kappa_0(a)t), \qquad t \in \mathbb{R},$$

and Z *is a standard normally distributed random variable independent of* $J_a(d)$.

4.6 Remark. If LAN property holds then one can construct asymptotically optimal tests, see, e.g., Theorem 15.4 and Addendum 15.5 of van der Vaart [9].

5 Maximum likelihood estimates

For fixed $T \in \mathbb{R}_{++}$, maximizing $\log\frac{\mathrm{d}\mathbb{P}_{a,T}}{\mathrm{d}\mathbb{P}_{0,T}}(X^{(a)}|_{[-1,T]})$ in $a \in \mathbb{R}$ gives the MLE of a based on the observations $(X(t))_{t\in[-1,T]}$ having the form

$$\widehat{a}_T = \frac{\int_0^T\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\,\mathrm{d}X^{(a)}(t)}{\int_0^T\left(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\right)^2\mathrm{d}t},$$

provided that $\int_0^T\left(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\right)^2\mathrm{d}t > 0$. Using the SDDE (1.1), one can check that

$$\widehat{a}_T - a = \frac{\int_0^T\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\,\mathrm{d}W(t)}{\int_0^T\left(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\right)^2\mathrm{d}t},$$

hence

$$r_{a,T}^{-1}(\widehat{a}_T - a) = \frac{\Delta_{a,T}}{J_{a,T}}.$$

Using the results of Section 4 and the continuous mapping theorem, we obtain the following result.

5.1 Proposition. *If* $a \in \left(-\frac{\pi^2}{2}, 0\right)$ *then*

$$\sqrt{T}\,(\widehat{a}_T - a) \xrightarrow{\mathcal{D}} \mathcal{N}(0, J_a^{-1}) \qquad \text{as } T \to \infty,$$

where J_a *is given in Proposition 4.1.*

If $a = 0$ *then*

$$T\,(\widehat{a}_T - a) = T\,\widehat{a}_T \stackrel{\mathcal{D}}{\longrightarrow} \frac{\int_0^1 \mathcal{W}(t)\,\mathrm{d}\mathcal{W}(t)}{\int_0^1 \mathcal{W}(t)^2\,\mathrm{d}t} \qquad \text{as } T \to \infty,$$

where $(\mathcal{W}(t))_{t\in[0,1]}$ *is a standard Wiener process.*

If $a = -\frac{\pi^2}{2}$ *then*

$$T\,(\widehat{a}_T - a) = T\left(\widehat{a}_T + \frac{\pi^2}{2}\right)$$

$$\stackrel{\mathcal{D}}{\longrightarrow} \frac{16\pi \int_0^1 (\mathcal{W}_1(t)\,\mathrm{d}\mathcal{W}_2(t) - \mathcal{W}_2(t)\,\mathrm{d}\mathcal{W}_1(t)) - 4\pi^2 \int_0^1 (\mathcal{W}_1(t)\,\mathrm{d}\mathcal{W}_1(t) + \mathcal{W}_2(t)\,\mathrm{d}\mathcal{W}_2(t))}{16 \int_0^1 (\mathcal{W}_1(t)^2 + \mathcal{W}_2(t)^2)\,\mathrm{d}t}$$

as $T \to \infty$, *where* $(\mathcal{W}_1(t), \mathcal{W}_2(t))_{t\in[0,1]}$ *is a 2-dimensional standard Wiener process.*

If $a \in (0, \infty)$ *then*

$$\mathrm{e}^{v_0(a)T}\,(\widehat{a}_T - a) \stackrel{\mathcal{D}}{\longrightarrow} \frac{Z}{\sqrt{J_a}} \qquad \text{as } T \to \infty,$$

where J_a *is given in Proposition 4.4, and* Z *is a standard normally distributed random variable independent of* J_a.

If $a \in \left(-\infty, -\frac{\pi^2}{2}\right)$ *then for each* $d \in \left[0, \frac{\pi}{\kappa_0(a)}\right)$,

$$\mathrm{e}^{v_0(a)(k\frac{\pi}{\kappa_0(a)}+d)}\,(\widehat{a}_{k\frac{\pi}{\kappa_0(a)}+d} - a) \stackrel{\mathcal{D}}{\longrightarrow} \frac{Z}{\sqrt{J_a(d)}} \qquad \text{as } k \to \infty,$$

where $J_a(d)$ *is given in Proposition 4.5, and* Z *is a standard normally distributed random variable independent of* $J_a(d)$.

If LAMN property holds then we have local asymptotic minimax bound for arbitrary estimators, see, e.g., Le Cam and Yang [6, 6.6, Theorem 1]. Maximum likelihood estimators attain this bound for bounded loss function, see, e.g., Le Cam and Yang [6, 6.6, Remark 11]. Moreover, maximum likelihood estimators are asymptotically efficient in Hájek's convolution theorem sense (see, for example, Le Cam and Yang [6, 6.6, Theorem 3 and Remark 13] or Jeganathan [5]).

6 Proofs

For each $a \in \mathbb{R}$ and each deterministic continuous function $(y(t))_{t\in\mathbb{R}_+}$, consider a continuous stochastic process $(Y^{(a)}(t))_{t\in\mathbb{R}_+}$ given by

$$Y^{(a)}(t) := y(t)X_0(0) + a\int_{-1}^0 \int_u^0 y(t+u-s)X_0(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t y(t-s)\,\mathrm{d}W(s) \tag{6.1}$$

for $t \in [1, \infty)$.

6.1 Lemma. *Let* $(y(t))_{t\in\mathbb{R}_+}$ *be a deterministic continuous function. Put*

$$Z(t) := \int_{-1}^{0}\int_{u}^{0} y(t+u-s)X_0(s)\,\mathrm{d}s\,\mathrm{d}u, \qquad t\in[1,\infty).$$

Then for each $T\in[1,\infty)$*,*

$$\int_1^T Z(t)^2\,\mathrm{d}t \leqslant \int_{-1}^{0} X_0(s)^2\,\mathrm{d}s \int_0^T y(v)^2\,\mathrm{d}v.$$

Proof. For each $t\in[1,\infty)$, by Fubini's theorem,

$$Z(t) = \int_{-1}^{0} X_0(s)\int_{-1}^{s} y(t+u-s)\,\mathrm{d}u\,\mathrm{d}s = \int_{-1}^{0} X_0(s)\int_{t-s-1}^{t} y(v)\,\mathrm{d}v\,\mathrm{d}s.$$

By the Cauchy–Schwarz inequality,

$$Z(t)^2 \leqslant \int_{-1}^{0} X_0(s)^2\,\mathrm{d}s \int_{-1}^{0}\left(\int_{t-s-1}^{t} y(v)\,\mathrm{d}v\right)^2\mathrm{d}s.$$

Consequently,

$$\int_1^T Z(t)^2\,\mathrm{d}t \leqslant \int_{-1}^{0} X_0(s)^2\,\mathrm{d}s \int_1^T\int_{-1}^{0}\left(\int_{t-s-1}^{t} y(v)\,\mathrm{d}v\right)^2\mathrm{d}s\,\mathrm{d}t,$$

where

$$\int_1^T\int_{-1}^{0}\left(\int_{t-s-1}^{t} y(v)\,\mathrm{d}v\right)^2\mathrm{d}s\,\mathrm{d}t = \int_{-1}^{0}\int_1^T\left(\int_{t-s-1}^{t} y(v)\,\mathrm{d}v\right)^2\mathrm{d}t\,\mathrm{d}s.$$

Here

$$\begin{aligned}\int_1^T\left(\int_{t-s-1}^{t} y(v)\,\mathrm{d}v\right)^2\mathrm{d}t &\leqslant \int_1^T\int_{t-s-1}^{t} y(v)^2\,\mathrm{d}v\,\mathrm{d}t = \int_{-s}^{T} y(v)^2\int_v^{v+s+1}\mathrm{d}t\,\mathrm{d}v\\ &\leqslant \int_{-s}^{T} y(v)^2\,\mathrm{d}v \leqslant \int_0^T y(v)^2\,\mathrm{d}v\end{aligned}$$

for all $s\in[-1,0]$, hence we obtain the statement. □

6.2 Lemma. *Let* $(y(t))_{t\in\mathbb{R}_+}$ *be a deterministic continuous function with* $\int_0^\infty y(t)^2\,\mathrm{d}t<\infty$*. Then for each* $a\in\mathbb{R}$*,*

$$\frac{1}{T}\int_0^T Y^{(a)}(t)\,\mathrm{d}t \xrightarrow{\mathbb{P}} 0 \qquad \text{as } T\to\infty,$$

$$\frac{1}{T}\int_0^T Y^{(a)}(t)^2\,\mathrm{d}t \xrightarrow{\mathbb{P}} \int_0^\infty y(t)^2\,\mathrm{d}t \qquad \text{as } T\to\infty.$$

Proof. Applying Lemma 4.3 of Gushchin and Küchler [1] for the special case $X_0(s) = 0$, $s \in [-1, 0]$, we obtain

$$\frac{1}{T}\int_0^T \int_0^t y(t-s)\,\mathrm{d}W(s)\,\mathrm{d}t \xrightarrow{\mathbb{P}} 0 \qquad \text{as } T \to \infty,$$

$$\frac{1}{T}\int_0^T \left(\int_0^t y(t-s)\,\mathrm{d}W(s)\right)^2 \mathrm{d}t \xrightarrow{\mathbb{P}} \int_0^\infty y(t)^2\,\mathrm{d}t \qquad \text{as } T \to \infty.$$

We have

$$\frac{1}{T}\int_0^T Y^{(a)}(t)\,\mathrm{d}t = \frac{1}{T}\int_0^1 Y^{(a)}(t)\,\mathrm{d}t + X_0(0)I_1(T) + \frac{a}{T}\int_1^T Z(t)\,\mathrm{d}t + \frac{1}{T}\int_1^T \int_0^t y(t-s)\,\mathrm{d}W(s)\,\mathrm{d}t$$

for $T \in \mathbb{R}_+$, where $(Z(t))_{t\in\mathbb{R}_+}$ is given in Lemma 6.1, and

$$I_1(T) := \frac{1}{T}\int_1^T y(t)\,\mathrm{d}t, \qquad T \in \mathbb{R}_+.$$

By Lemma 6.1,

$$|I_1(T)| \leqslant \sqrt{\int_0^T \frac{1}{T^2}\,\mathrm{d}t \int_1^T y(t)^2\,\mathrm{d}t} = \sqrt{\frac{1}{T}\int_0^\infty y(t)^2\,\mathrm{d}t} \to 0 \qquad \text{as } T \to \infty,$$

$$\left|\frac{1}{T}\int_1^T Z(t)\,\mathrm{d}t\right| \leqslant \sqrt{\frac{1}{T}\int_1^T Z(t)^2\,\mathrm{d}t} \leqslant \sqrt{\frac{1}{T}\int_{-1}^0 X_0(s)^2\,\mathrm{d}s \int_0^\infty y(v)^2\,\mathrm{d}v} \xrightarrow{\text{a.s.}} 0 \qquad \text{as } T \to \infty,$$

hence we obtain the first statement. Moreover,

$$\frac{1}{T}\int_0^T Y^{(a)}(t)^2\,\mathrm{d}t = \frac{1}{T}\int_0^1 Y^{(a)}(t)^2\,\mathrm{d}t + I_2(T) + 2I_3(T) + \frac{1}{T}\int_1^T \left(\int_0^t y(t-s)\,\mathrm{d}W(s)\right)^2 \mathrm{d}t$$

for $T \in \mathbb{R}_+$, where

$$I_2(T) := \frac{1}{T}\int_1^T (y(t)X_0(0) + aZ(t))^2\,\mathrm{d}t, \qquad T \in \mathbb{R}_+,$$

$$I_3(T) := \frac{1}{T}\int_1^T (y(t)X_0(0) + aZ(t))\left(\int_0^t y(t-s)\,\mathrm{d}W(s)\right)\mathrm{d}t, \qquad T \in \mathbb{R}_+.$$

Again by Lemma 6.1,

$$\begin{aligned} 0 \leqslant I_2(T) &\leqslant \frac{1}{T}\int_1^T 2(y(t)^2X_0(0)^2 + a^2Z(t)^2)\,\mathrm{d}t \\ &\leqslant \frac{2X_0(0)^2}{T}\int_0^\infty y(t)^2\,\mathrm{d}t + \frac{2a^2}{T}\int_{-1}^0 X_0(s)^2\,\mathrm{d}s \int_0^\infty y(v)^2\,\mathrm{d}v \xrightarrow{\text{a.s.}} 0 \qquad \text{as } T \to \infty, \end{aligned}$$

and

$$|I_3(T)| \leqslant \frac{2}{T}\sqrt{\int_1^T (y(t)^2 X_0(0)^2 + a^2 Z(t)^2)^2 \,\mathrm{d}t \int_1^T \left(\int_0^t y(t-s)\,\mathrm{d}W(s)\right)^2 \mathrm{d}t}$$

$$= 2\sqrt{\frac{I_2(T)}{T}\int_1^T \left(\int_0^t y(t-s)\,\mathrm{d}W(s)\right)^2 \mathrm{d}t} \xrightarrow{\mathbb{P}} 0 \qquad \text{as } T \to \infty,$$

hence we obtain the second statement. □

6.3 Lemma. *Let $w \in \mathbb{R}_{++}$ and $y(t) := \mathrm{e}^{wt}$, $t \in \mathbb{R}_+$. Then for each $a \in \mathbb{R}$,*

$$\mathrm{e}^{-wt} Y^{(a)}(t) \xrightarrow{\text{a.s.}} U_w^{(a)}, \qquad \textit{as } t \to \infty,$$

$$\mathrm{e}^{-2wT} \int_0^T (Y^{(a)}(t))^2 \,\mathrm{d}t \xrightarrow{\text{a.s.}} \frac{1}{2w}(U_w^{(a)})^2, \qquad \textit{as } T \to \infty,$$

with

$$U_w^{(a)} := X_0(0) + a\int_{-1}^0 \int_u^0 \mathrm{e}^{w(u-s)} X_0(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^\infty \mathrm{e}^{-ws}\,\mathrm{d}W(s).$$

Proof. For each $t \in [1, \infty)$, we have

$$\mathrm{e}^{-wt} Y^{(a)}(t) = X_0(0) + a\int_{-1}^0 \int_u^0 \mathrm{e}^{w(u-s)} X_0(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t \mathrm{e}^{-ws}\,\mathrm{d}W(s),$$

hence we obtain the first convergence. The second convergence follows by L'Hôspital's rule. □

6.4 Lemma. *Let $w \in \mathbb{R}_{++}$, $\kappa \in \mathbb{R}$, and $y(t) := \varphi(t)\mathrm{e}^{wt}$, $t \in \mathbb{R}_+$, with $\varphi(t) = \cos(\kappa t)$, $t \in \mathbb{R}_+$, or $\varphi(t) = \sin(\kappa t)$, $t \in \mathbb{R}_+$. Then for each $a \in \mathbb{R}$,*

$$\mathrm{e}^{-wt} Y^{(a)}(t) - V_w^{(a)}(t) \xrightarrow{\text{a.s.}} 0, \qquad \textit{as } t \to \infty,$$

$$\mathrm{e}^{-2wT} \int_0^T (Y^{(a)}(t))^2 \,\mathrm{d}t - \int_0^\infty \mathrm{e}^{-2wt} (V_w^{(a)}(T-t))^2 \,\mathrm{d}t \xrightarrow{\mathbb{P}} 0, \qquad \textit{as } T \to \infty,$$

with

$$V_w^{(a)}(t) := X_0(0)\varphi(t) + a\int_{-1}^0 \int_u^0 \varphi(t+u-s)\mathrm{e}^{w(u-s)} X_0(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^\infty \varphi(t-s)\mathrm{e}^{-ws}\,\mathrm{d}W(s)$$

for $t \in \mathbb{R}$.

Proof. Note that for each $t \in [1, \infty)$,

$$\mathrm{e}^{-wt} Y^{(a)}(t) - V_w^{(a)}(t) = -\int_t^\infty \varphi(t-s)\mathrm{e}^{-ws}\,\mathrm{d}W(s), \tag{6.2}$$

which obviously tends almost surely to zero as $t \to \infty$, hence we obtain the first convergence.

In order to prove the second convergence, observe that for each $T \in \mathbb{R}_+$,

$$\int_0^\infty \mathrm{e}^{-2wt}(V_w^{(a)}(T-t))^2 \,\mathrm{d}t = \int_0^T \mathrm{e}^{-2wt}(V_w^{(a)}(T-t))^2 \,\mathrm{d}t + \int_T^\infty \mathrm{e}^{-2wt}(V_w^{(a)}(T-t))^2 \,\mathrm{d}t,$$

where

$$\int_0^T \mathrm{e}^{-2wt}(V_w^{(a)}(T-t))^2 \,\mathrm{d}t = \int_0^T \mathrm{e}^{-2w(T-t)}(V_w^{(a)}(t))^2 \,\mathrm{d}t = \mathrm{e}^{-2wT}\int_0^T (\mathrm{e}^{wt}V_w^{(a)}(t))^2 \,\mathrm{d}t,$$

hence

$$\begin{aligned}
&\mathrm{e}^{-2wT}\int_0^T (Y^{(a)}(t))^2 \,\mathrm{d}t - \int_0^\infty \mathrm{e}^{-2wt}(V_w^{(a)}(T-t))^2 \,\mathrm{d}t \\
&= \mathrm{e}^{-2wT}\int_0^T \big[(Y^{(a)}(t))^2 - (\mathrm{e}^{wt}V_w^{(a)}(t))^2\big]\mathrm{d}t - \int_T^\infty \mathrm{e}^{-2wt}(V_w^{(a)}(T-t))^2 \,\mathrm{d}t \\
&= I_0(T) + I_1(T) + 2I_2(T) - I_3(T)
\end{aligned}$$

with

$$\begin{aligned}
I_0(T) &:= \mathrm{e}^{-2wT}\int_0^1 (Y^{(a)}(t) - \mathrm{e}^{wt}V_w^{(a)}(t))^2 \,\mathrm{d}t, \\
I_1(T) &:= \mathrm{e}^{-2wT}\int_1^T (Y^{(a)}(t) - \mathrm{e}^{wt}V_w^{(a)}(t))^2 \,\mathrm{d}t, \\
I_2(T) &:= \mathrm{e}^{-2wT}\int_0^T (Y^{(a)}(t) - \mathrm{e}^{wt}V_w^{(a)}(t))\mathrm{e}^{wt}V_w^{(a)}(t) \,\mathrm{d}t, \\
I_3(T) &:= \int_T^\infty \mathrm{e}^{-2wt}(V_w^{(a)}(T-t))^2 \,\mathrm{d}t.
\end{aligned}$$

The processes $(Y^{(a)}(t))_{t\in\mathbb{R}_+}$ and $(V_w^{(a)}(t))_{t\in\mathbb{R}_+}$ are continuous, hence

$$\mathbb{E}(|I_0(T)|) = \mathrm{e}^{-2wT}\int_0^1 \mathbb{E}\big[(Y^{(a)}(t) - \mathrm{e}^{wt}V_w^{(a)}(t))^2\big]\mathrm{d}t \to 0 \qquad \text{as } T \to \infty,$$

implying $I_0(T) \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$. By (6.2),

$$I_1(T) = \mathrm{e}^{-2wT}\int_1^T \mathrm{e}^{2wt}\left(\int_t^\infty \varphi(t-s)\mathrm{e}^{-ws}\,\mathrm{d}W(s)\right)^2 \mathrm{d}t,$$

hence

$$\begin{aligned}
\mathbb{E}(|I_1(T)|) &= \mathrm{e}^{-2wT}\int_1^T \mathrm{e}^{2wt}\int_t^\infty \varphi(t-s)^2\mathrm{e}^{-2ws}\,\mathrm{d}s\,\mathrm{d}t \leqslant \mathrm{e}^{-2wT}\int_1^T \mathrm{e}^{2wt}\int_t^\infty \mathrm{e}^{-2ws}\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{T}{2w}\mathrm{e}^{-2wT} \to 0 \qquad \text{as } T \to \infty,
\end{aligned}$$

implying $I_1(T) \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$. Moreover, by the Cauchy–Schwarz inequality,

$$|I_2(T)| \leqslant \sqrt{I_1(T)\mathrm{e}^{-2wT} \int_0^T (\mathrm{e}^{wt} V_w^{(a)}(t))^2 \,\mathrm{d}t}$$

with

$$\mathrm{e}^{-2wT} \int_0^T (\mathrm{e}^{wt} V_w^{(a)}(t))^2 \,\mathrm{d}t = \int_0^T \mathrm{e}^{-2wt} V_w^{(a)}(T-t)^2 \,\mathrm{d}t \leqslant \frac{1}{2w} \sup_{t\in\mathbb{R}} (V_w^{(a)}(t))^2,$$

where $\sup_{t\in\mathbb{R}} (V_w^{(a)}(t))^2 < \infty$ almost surely, since $(V_w^{(a)}(t))_{t\in\mathbb{R}}$ is a continuous and periodic process. Consequently, $I_2(T) \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$. Finally,

$$|I_3(T)| \leqslant \frac{\mathrm{e}^{-2wT}}{2w} \sup_{t\in\mathbb{R}} (V_w^{(a)}(t))^2 \xrightarrow{\text{a.s.}} 0 \qquad \text{as } T \to \infty,$$

hence we obtain the second convergence of the statement. $\square$

Proof of Proposition 4.1. For each $t \in [1, \infty)$, by (1.4), we have

$$\begin{aligned}\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u &= X_0(0)\int_{-1}^0 x_{0,a}(t+u)\,\mathrm{d}u + a\int_{-1}^0\int_{-1}^0\int_v^0 x_{0,a}(t+u+v-s)X_0(s)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}u \\ &\quad + \int_{-1}^0\int_0^{t+u} x_{0,a}(t+u-s)\,\mathrm{d}W(s)\,\mathrm{d}u.\end{aligned}$$

Here we have

$$\begin{aligned}\int_{-1}^0\int_{-1}^0\int_v^0 x_{0,a}(t+u+v-s)X_0(s)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}u &= \int_{-1}^0\int_{-1}^0\int_v^0 x_{0,a}(t+u+v-s)X_0(s)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{-1}^0\int_v^0 X_0(s)\int_{-1}^0 x_{0,a}(t+u+v-s)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v,\end{aligned}$$

and

$$\begin{aligned}&\int_{-1}^0\int_0^{t+u} x_{0,a}(t+u-s)\,\mathrm{d}W(s)\,\mathrm{d}u \\ &= \int_0^{t-1}\int_{-1}^0 x_{0,a}(t+u-s)\,\mathrm{d}u\,\mathrm{d}W(s) + \int_{t-1}^t\int_{s-t}^0 x_{0,a}(t+u-s)\,\mathrm{d}u\,\mathrm{d}W(s) \\ &= \int_0^t\int_{-1}^0 x_{0,a}(t+u-s)\,\mathrm{d}u\,\mathrm{d}W(s),\end{aligned}$$

since $t \in [1, \infty)$, $s \in [t-1, t]$ and $u \in [-1, s-t)$ imply $t+u-s \in [-1, 0)$, and hence $x_{0,a}(t+u-s) = 0$. Consequently, the process $\left(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\right)_{t\in\mathbb{R}_+}$ has a representation (6.1) with

$$y(t) = \int_{-1}^0 x_{0,a}(t+u)\,\mathrm{d}u, \qquad t \in \mathbb{R}_+.$$

Assumption $a \in \left(-\frac{\pi^2}{2}, 0\right)$ implies $v_0(a) < 0$, and hence $\int_0^\infty x_{0,a}(t)^2 \, \mathrm{d}t < \infty$ holds. Thus

$$\int_1^\infty y(t)^2 \, \mathrm{d}t = \int_1^\infty \left(\int_{-1}^0 x_{0,a}(t+u) \, \mathrm{d}u \right)^2 \mathrm{d}t$$
$$= \int_{-1}^0 \int_{-1}^0 \int_1^\infty x_{0,a}(t+u) x_{0,a}(t+v) \, \mathrm{d}t \, \mathrm{d}u \, \mathrm{d}v \leqslant \int_0^\infty x_{0,a}(t)^2 \, \mathrm{d}t,$$

since

$$\left| \int_1^\infty x_{0,a}(t+u) x_{0,a}(t+v) \, \mathrm{d}t \right| \leqslant \sqrt{\int_1^\infty x_{0,a}(t+u)^2 \, \mathrm{d}t \int_1^\infty x_{0,a}(t+v)^2 \, \mathrm{d}t}$$
$$= \sqrt{\int_{1+u}^\infty x_{0,a}(s+u)^2 \, \mathrm{d}s \int_{1+v}^\infty x_{0,a}(s+v)^2 \, \mathrm{d}s} \leqslant \int_0^\infty x_{0,a}(t)^2 \, \mathrm{d}t.$$

Consequently, $\int_0^\infty y(t)^2 \, \mathrm{d}t \leqslant \int_0^1 y(t)^2 \, \mathrm{d}t + \int_0^\infty x_{0,a}(t)^2 \, \mathrm{d}t < \infty$, thus we can apply Lemma 6.2 to obtain

$$J_{a,T} = \frac{1}{T} \int_0^T \left(\int_{-1}^0 X^{(a)}(t+u) \, \mathrm{d}u \right)^2 \mathrm{d}t \xrightarrow{\mathbb{P}} \int_0^\infty \left(\int_{-1}^0 x_{0,a}(t+u) \, \mathrm{d}u \right)^2 \mathrm{d}t = J_a$$

as $T \to \infty$. Moreover, the process

$$M^{(a)}(T) := \int_0^T \int_{-1}^0 X^{(a)}(t+u) \, \mathrm{d}u \, \mathrm{d}W(t), \qquad T \in \mathbb{R}_+,$$

is a continuous martingale with $M^{(a)}(0) = 0$ and with quadratic variation

$$\langle M^{(a)} \rangle (T) = \int_0^T \left(\int_{-1}^0 X^{(a)}(t+u) \, \mathrm{d}u \right)^2 \mathrm{d}t,$$

hence, Theorem VIII.5.42 of Jacod and Shiryaev [4] yields the statement. □

Proof of Proposition 4.2. We have

$$\Delta_{0,T} = \frac{1}{T} \int_0^T \int_{-1}^0 X^{(0)}(t+u) \, \mathrm{d}u \, \mathrm{d}W(t), \qquad T \in \mathbb{R}_{++}.$$

As in the proof of Proposition 4.1, for each $t \in [1, \infty)$, we obtain

$$\int_{-1}^0 X^{(0)}(t+u) \, \mathrm{d}u = X_0(0) \int_{-1}^0 x_{0,0}(t+u) \, \mathrm{d}u + \int_0^t \int_{-1}^0 x_{0,0}(t+u-s) \, \mathrm{d}u \, \mathrm{d}W(s).$$

Here we have

$$\int_{-1}^0 x_{0,0}(t+u) \, \mathrm{d}u = 1, \qquad \int_{-1}^0 x_{0,0}(t+u-s) \, \mathrm{d}u = \begin{cases} 1, & \text{for } s \in [0, t-1], \\ t-s, & \text{for } s \in [t-1, t], \end{cases}$$

hence

$$\int_{-1}^{0} X^{(0)}(t+u)\,\mathrm{d}u = X_0(0) + \int_0^{t-1} \mathrm{d}W(s) + \int_{t-1}^{t} (t-s)\,\mathrm{d}W(s)$$

$$= X_0(0) + W(t) + \int_{t-1}^{t} (t-s-1)\,\mathrm{d}W(s) = W(t) + \overline{X}(t),$$

where $\mathbb{E}\big(T^{-2}\int_0^T \overline{X}(t)^2\,\mathrm{d}t\big) \to 0$ as $T \to \infty$. For each $T \in \mathbb{R}_{++}$, consider the process

$$W^T(s) := \frac{1}{\sqrt{T}} W(Ts), \qquad s \in [0,1].$$

Then we have

$$\Delta_{0,T} = \int_0^1 W^T(t)\,\mathrm{d}W^T(t) + \frac{1}{T}\int_0^T \overline{X}(t)\,\mathrm{d}W(t),$$

$$J_{0,T} = \int_0^1 W^T(t)^2\,\mathrm{d}t + \frac{2}{T^2}\int_0^T W(t)\overline{X}(t)\,\mathrm{d}t + \frac{1}{T^2}\int_0^T \overline{X}(t)^2\,\mathrm{d}t.$$

Here

$$\frac{1}{T}\int_0^T \overline{X}(t)\,\mathrm{d}W(t) \xrightarrow{\mathbb{P}} 0, \qquad \frac{1}{T^2}\int_0^T \overline{X}(t)^2\,\mathrm{d}t \xrightarrow{\mathbb{P}} 0$$

as $T \to \infty$, since

$$\mathbb{E}\left[\left(\frac{1}{T}\int_0^T \overline{X}(t)\,\mathrm{d}W(t)\right)^2\right] = \frac{1}{T^2}\int_0^T \mathbb{E}(\overline{X}(t)^2)\,\mathrm{d}t \to 0.$$

By the functional central limit theorem,

$$W^T \xrightarrow{\mathcal{D}} \mathcal{W} \qquad \text{as } T \to \infty,$$

hence

$$\left|\frac{1}{T^2}\int_0^T W(t)\overline{X}(t)\,\mathrm{d}t\right| \leqslant \sqrt{\left(\frac{1}{T^2}\int_0^T W(t)^2\,\mathrm{d}t\right)\left(\frac{1}{T^2}\int_0^T \overline{X}(t)^2\,\mathrm{d}t\right)}$$

$$= \sqrt{\left(\int_0^1 W^T(t)^2\,\mathrm{d}t\right)\left(\frac{1}{T^2}\int_0^T \overline{X}(t)^2\,\mathrm{d}t\right)} \xrightarrow{\mathbb{P}} 0 \qquad \text{as } T \to \infty,$$

and the claim follows from Corollary 4.12 in Gushchin and Küchler [1]. □

Proof of Proposition 4.3. We have

$$\Delta_{-\frac{\pi^2}{2},T} = \frac{1}{T}\int_0^T \int_{-1}^0 X^{(-\pi^2/2)}(t+u)\,\mathrm{d}u\,\mathrm{d}W(t), \qquad T \in \mathbb{R}_{++},$$

$$J_{-\frac{\pi^2}{2},T} = \frac{1}{T^2}\int_0^T \left(\int_{-1}^0 X^{(-\pi^2/2)}(t+u)\,\mathrm{d}u\right)^2 \mathrm{d}t, \qquad T \in \mathbb{R}_{++}.$$

As in the proof of Proposition 4.1, for each $t \in [1, \infty)$, we have

$$\int_{-1}^{0} X^{(-\pi^2/2)}(t+u)\,\mathrm{d}u = X_0(0)\int_{-1}^{0} x_{0,-\frac{\pi^2}{2}}(t+u)\,\mathrm{d}u + \int_0^t \int_{-1}^{0} x_{0,-\frac{\pi^2}{2}}(t+u-s)\,\mathrm{d}u\,\mathrm{d}W(s)$$
$$- \frac{\pi^2}{2}\int_{-1}^{0}\int_{v}^{0} X_0(s)\int_{-1}^{0} x_{0,-\frac{\pi^2}{2}}(t+u+v-s)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v.$$

We have $v_0\big(-\frac{\pi^2}{2}\big) = 0$ and $\kappa_0\big(-\frac{\pi^2}{2}\big) = \pi$, hence $A_0\big(-\frac{\pi^2}{2}\big) = \frac{16}{\pi^2+16}$ and $B_0\big(-\frac{\pi^2}{2}\big) = \frac{4\pi}{\pi^2+16}$. Consequently, by Lemma 1.1, there exists $\gamma \in (-\infty, 0)$ such that

$$x_{0,-\frac{\pi^2}{2}}(t) = \frac{16\cos(\pi t) + 4\pi\sin(\pi t)}{\pi^2+16} + \mathrm{o}(\mathrm{e}^{\gamma t}), \qquad \text{as } t \to \infty,$$

and hence

$$\int_{-1}^{0} X^{(-\pi^2/2)}(t+u)\,\mathrm{d}u = \int_0^t \int_{-1}^{0} \frac{16\cos(\pi(t+u-s)) + 4\pi\sin(\pi(t+u-s))}{\pi^2+16}\,\mathrm{d}u\,\mathrm{d}W(s) + \overline{X}(t)$$
$$= \int_0^t \frac{32\sin(\pi(t-s)) - 8\pi\cos(\pi(t-s))}{\pi(\pi^2+16)}\,\mathrm{d}W(s) + \overline{X}(t),$$

where $T^{-2}\int_0^T \overline{X}(t)^2\,\mathrm{d}t \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$. Introducing

$$X_1(t) := \int_0^t \cos(\pi s)\,\mathrm{d}W(s), \qquad X_2(t) := \int_0^t \sin(\pi s)\,\mathrm{d}W(s), \qquad t \in \mathbb{R}_+,$$

we obtain

$$\int_{-1}^{0} X^{(-\pi^2/2)}(t+u)\,\mathrm{d}u$$
$$= \frac{32X_1(t)\sin(\pi t) - 32X_2(t)\cos(\pi t) - 8\pi X_1(t)\cos(\pi t) - 8\pi X_2(t)\sin(\pi t)}{\pi(\pi^2+16)} + \overline{X}(t).$$

For each $T \in \mathbb{R}_{++}$, consider the following processes on $[0,1]$:

$$W^T(s) := \frac{1}{\sqrt{T}}W(Ts),$$
$$X_1^T(s) := \frac{1}{\sqrt{T}}X_1(Ts) = \int_0^s \cos(\pi Ts)\,\mathrm{d}W^T(s),$$
$$X_2^T(s) := \frac{1}{\sqrt{T}}X_2(Ts) = \int_0^s \sin(\pi Ts)\,\mathrm{d}W^T(s),$$
$$X^T(s) := \frac{32X_1^T(s)\sin(\pi Ts) - 32X_2^T(s)\cos(\pi Ts) - 8\pi X_1(s)\cos(\pi Ts) - 8\pi X_2(s)\sin(\pi Ts)}{\pi(\pi^2+16)}.$$

Then, for each $T \in \mathbb{R}_{++}$, we have

$$\int_{-1}^{0} X^{(-\pi^2/2)}(t+u)\,\mathrm{d}u = \sqrt{T}X^T\Big(\frac{t}{T}\Big) + \overline{X}(t),$$

and hence,

$$\Delta_{-\frac{\pi^2}{2},T} = \frac{1}{\sqrt{T}} \int_0^T X^T\Big(\frac{t}{T}\Big) \,\mathrm{d}W(t) + I_1(T) = \int_0^1 X^T(s) \,\mathrm{d}W^T(s) + I_1(T),$$

$$J_{-\frac{\pi^2}{2},T} = \frac{1}{T} \int_0^T X^T\Big(\frac{t}{T}\Big)^2 \mathrm{d}t + 2I_2(T) + I_3(T) = \int_0^1 X^T(s)^2 \,\mathrm{d}s + 2I_2(T) + I_3(T),$$

with

$$I_1(T) := \frac{1}{T} \int_0^T \overline{X}(t) \,\mathrm{d}W(t), \quad I_2(T) := \frac{1}{T^{3/2}} \int_0^T X^T\Big(\frac{t}{T}\Big)\overline{X}(t) \,\mathrm{d}t, \quad I_3(T) := \frac{1}{T^2} \int_0^T \overline{X}(t)^2 \,\mathrm{d}t.$$

Introducing the process

$$Y^T(t) := \int_0^t X^T(s) \,\mathrm{d}W^T(s), \qquad t \in \mathbb{R}_+, \qquad T \in \mathbb{R}_{++},$$

we have

$$\int_0^t X^T(s)^2 \,\mathrm{d}s = [Y^T, Y^T](t), \qquad t \in \mathbb{R}_+, \qquad T \in \mathbb{R}_{++},$$

where $([U,V](t))_{t\in\mathbb{R}_+}$ denotes the quadratic covariation process of the processes $(U(t))_{t\in\mathbb{R}_+}$ and $(V(t))_{t\in\mathbb{R}_+}$. Moreover,

$$Y^T(t) = \frac{32\int_0^t (X_1^T(s)\,\mathrm{d}X_2^T(s) - X_2^T(s)\,\mathrm{d}X_1^T(s)) - 8\pi\int_0^t (X_1^T(s)\,\mathrm{d}X_1^T(s) + X_2^T(s)\,\mathrm{d}X_2^T(s))}{\pi(\pi^2+16)}$$

for $t \in \mathbb{R}_+$. By the functional central limit theorem,

$$(X_1^T, X_2^T) \stackrel{\mathcal{D}}{\longrightarrow} \frac{1}{\sqrt{2}}(\mathcal{W}_1, \mathcal{W}_2) \qquad \text{as } T \to \infty,$$

hence

$$Y^T \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{Y} \qquad \text{as } T \to \infty$$

with

$$\mathcal{Y}(t) := \frac{16\int_0^t (\mathcal{W}_1(s)\,\mathrm{d}\mathcal{W}_2(s) - \mathcal{W}_2(s)\,\mathrm{d}\mathcal{W}_1(s)) - 4\pi\int_0^t (\mathcal{W}_1(s)\,\mathrm{d}\mathcal{W}_1(s) + \mathcal{W}_2(s)\,\mathrm{d}\mathcal{W}_2(s))}{\pi(\pi^2+16)}$$

for $t \in \mathbb{R}_+$. Further, by Corollary 4.12 in Gushchin and Küchler [1],

$$(Y^T(1), [Y^T, Y^T](1)) \stackrel{\mathcal{D}}{\longrightarrow} (\mathcal{Y}(1), [\mathcal{Y}, \mathcal{Y}](1)) \qquad \text{as } T \to \infty.$$

Here we have

$$\begin{aligned}
[\mathcal{Y}, \mathcal{Y}](1) &= \frac{\int_0^1 (16\mathcal{W}_1(s) - 4\pi\mathcal{W}_2(s))^2 \mathrm{d}s + \int_0^1 (16\mathcal{W}_2(s) + 4\pi\mathcal{W}_1(s))^2 \mathrm{d}s}{\pi^2(\pi^2+16)^2} \\
&= \frac{16}{\pi^2(\pi^2+16)} \int_0^1 (\mathcal{W}_1(s)^2 + \mathcal{W}_2(s)^2) \,\mathrm{d}s.
\end{aligned}$$

Recall that $I_3(T) \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$, which also implies $I_1(T) \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$. Finally,

$$|I_2(T)| \leqslant \sqrt{\frac{1}{T^3}\int_0^T X^T\Big(\frac{t}{T}\Big)^2 \mathrm{d}t \int_0^T \overline{X}(t)^2\,\mathrm{d}t} = \sqrt{\frac{1}{T^2}\int_0^1 X^T(s)^2\,\mathrm{d}s \int_0^T \overline{X}(t)^2\,\mathrm{d}t} \xrightarrow{\mathbb{P}} 0$$

as $T \to \infty$, and the claim follows. □

Proof of Proposition 4.4. We have

$$J_{a,T} = \mathrm{e}^{-2v_0(a)T}\int_0^T \left(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\right)^2 \mathrm{d}t \qquad T \in \mathbb{R}_+.$$

The process $\big(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\big)_{t\in[1,\infty)}$ has a representation (6.1) with $y(t) = \int_{-1}^0 x_{0,a}(t+u)\,\mathrm{d}u$, $t \in \mathbb{R}_+$, see the proof of Proposition 4.1. The assumption $a \in (0,\infty)$ implies $v_0(a) > 0$ and $v_1(a) < 0$, hence by Lemma 1.1, there exists $\gamma \in (v_1(a), 0)$ such that

$$x_{0,a}(t) = \frac{v_0(a)}{v_0(a)^2 + 2v_0(a) - a}\,\mathrm{e}^{v_0(a)t} + \mathrm{o}(\mathrm{e}^{\gamma t}), \qquad \text{as } t \to \infty.$$

Consequently,

$$\int_{-1}^0 x_{0,a}(t+u)\,\mathrm{d}u = \frac{1-\mathrm{e}^{-v_0(a)}}{v_0(a)^2 + 2v_0(a) - a}\,\mathrm{e}^{v_0(a)t} + \mathrm{o}(\mathrm{e}^{\gamma t}), \qquad \text{as } t \to \infty.$$

Applying Lemma 6.3, we obtain

$$J_{a,T} \xrightarrow{\mathbb{P}} \frac{1}{2v_0(a)}\left(\frac{1-\mathrm{e}^{-v_0(a)}}{v_0(a)^2 + 2v_0(a) - a}\right)^2 (U^{(a)})^2 = J_a \qquad \text{as } T \to \infty.$$

Theorem VIII.5.42 of Jacod and Shiryaev [4] yields the statement. □

Proof of Proposition 4.5. We have again

$$J_{a,T} = \mathrm{e}^{-2v_0(a)T}\int_0^T \left(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\right)^2 \mathrm{d}t \qquad T \in \mathbb{R}_+,$$

and the process $\big(\int_{-1}^0 X^{(a)}(t+u)\,\mathrm{d}u\big)_{t\in[1,\infty)}$ has a representation (6.1) with $y(t) = \int_{-1}^0 x_{0,a}(t+u)\,\mathrm{d}u$, $t \in \mathbb{R}_+$, see the proof of Proposition 4.1. The assumption $a \in \big(-\infty, -\frac{\pi^2}{2}\big)$ implies $v_0(a) > 0$ and $v_0(a) \notin \Lambda_a$, hence by Lemma 1.1, there exists $\gamma \in (0, v_0(a))$ such that

$$x_{0,a}(t) = \varphi_a(t)\mathrm{e}^{v_0(a)t} + \mathrm{o}(\mathrm{e}^{\gamma t}), \qquad \text{as } t \to \infty.$$

Applying Lemma 6.4, we obtain

$$J_{a,T} - J_a(T) \xrightarrow{\mathbb{P}} 0, \qquad \text{as } T \to \infty.$$

The process $(J_a(t))_{t\in\mathbb{R}_+}$ is periodic with period $D = \frac{\pi}{\kappa_0(a)}$. □

References

[1] Gushchin, A. A. and Küchler, U. (1999). Asymptotic inference for a linear stochastic differential equation with time delay. *Bernoulli* **5(6)** 1059–1098.

[2] Gushchin, A. A. and Küchler, U. (2000). On stationary solutions of delay differential equations driven by a Lévy process. *Stochastic Processes and their Applications* **88(2)** 195–211.

[3] Gushchin, A. A. and Küchler, U. (2003). On parametric statistical models for stationary solutions of affine stochastic delay differential equations. *Mathematical Methods of Statistics* **12(1)** 31–61.

[4] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. Springer-Verlag, Berlin.

[5] Jeganathan, P. (1995). Some aspects of asymptotic theory with applications to time series models. *Econometric Theory* **11(5)** 818–887.

[6] Le Cam, L. and Yang, G. L. (2000). *Asymptotics in statistics: some basic concepts*, Springer.

[7] Liptser, R. S. and Shiryaev, A. N. (2001). *Statistics of Random Processes I. Applications*, 2nd edition. Springer-Verlag, Berlin, Heidelberg.

[8] Reiss, M. (2002). *Nonparametric estimation for stochastic delay differential equations.* PhD thesis, Humboldt-Universität, Mathematisch-Naturwissenschaftliche Fakultät II, 123 S., Berlin.
`http://dochost.rz.hu-berlin.de/dissertationen/reiss-markus-2002-02-13/`

[9] van der Vaart, A. W. (1998). *Asymptotic Statistics*, Cambridge University Press.